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# CONTACT OF ONE OR MORE SLENDER BODIES WITH AN ELASTIC HALF SPACE

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**Abstract**—Integral equations are determined relating force and displacement distributions along slender bodies lying in or on an elastic half space. Every such body is shown to be sectionally equivalent to some flat frictionless punch, and procedures are given for computing the parameters of that equivalent punch, for various actual contact states. The resulting integral equation is a generalization of one first derived by Kalker, and new results are given for the force distributions on footings of various types, including cases of interaction between two or more adjacent bodies.

## 1. INTRODUCTION

The mechanics of contact between two media, one or both being elastic, is of long-standing interest in solid mechanics and engineering. Mathematically, these problems pose challenges, because they involve different types of boundary conditions over different parts of the media. From the recent comprehensive surveys by Kalker [1] and Gladwell [2], it appears that two-dimensional, or plane-strain, problems have received the most thorough analytical investigation, progress being due significantly to the use of the powerful techniques of complex analysis [3].

Three-dimensional problems have been much less studied; exact solutions are often limited to relatively-simple geometries, such as circular or elliptical punches with a flat frictionless base. General geometries can only be handled by strictly numerical means, involving boundary integral equations in two independent variables, or three-dimensional finite elements.

An outstanding exception is the work of Kalker [4, 5] who treated the important case of contact of a slender (i.e. elongated) body with an elastic half-space, by asymptotic methods that were well-known in aerodynamics [6, 7]. In essence, this method treats two-dimensionally the neighbourhood of each separate cross-section of the body, within a distance of the order of $a$ from its longitudinal axis, where $2a$, the maximum width, is much less than $2l$, the length. Although such a strictly two-dimensional theory yields acceptable stresses, the two-dimensional displacement due to a prescribed force per unit length is indeterminate; indeed, it becomes unbounded logarithmically at infinity. An outer approximation valid at distances comparable to $l$, which treats the body as a distribution of surface traction along the axis of the body, must be matched to yield finite displacement everywhere. The final mathematical task is reduced by the slender-body approximation to the solution of an integral equation along the axis, in just one independent variable. Numerical solutions of this equation have been given for a flat rectangular frictionless punch by Panek and Kalker [8]. A similar theory has also been given by Sivashinsky [9].

In the present article, we first extend this slender-body approach, to allow in principle any type of contact along any boundary curve in the inner region. We introduce two fundamental parameters $a_0$ and $v_0$, each of which is obtainable by analytic or numerical solution of a suitably-normalized plane-strain problem, at each separate cross-section. These parameters are defined in terms of an "*equivalent frictionless flat punch*". That is, irrespective of how or where contact is actually made between body and elastic medium, the resulting plane-strain disturbance appears in its own far field as if it had been created by a uniform vertical displacement $v_0$, of a segment of width $2a_0$ of an originally-undisturbed plane free surface, with vanishing shear stress over that segment.

The integral equation resulting from matching this inner result to a line of three-dimensional surface tractions, is identical to that used by Panek and Kalker [8]. We provide here further discussion of the properties of this integral equation, which reproduces known exact results for elliptical plan forms, and allows rapidly-converging numerical solution, for a wide variety of smooth-ended bodies. Certain difficulties (already noticed by Panek and Kalker [8]) occur for extremely blunt-ended plan forms, such as the rectangular case $a_0 = \text{const}$. Although further work is needed to resolve these problems, adequate accuracy is achieved for integrated quantities such as the overall compliance of the body, by replacing any actual blunt-ended plan-form by one with smoother (e.g. elliptical) ends, but having the same total effective area.

Since many contact problems important to foundation engineering [10, 11] involve elongated footings which are not far apart, the interaction between them due to the elastic ground is of vital interest. One of the main purposes of this paper is to extend the asymptotic method to the case of several slender bodies. The type of analysis needed is similar to that used in other areas of application, such as aerodynamics of bi-plane wings, and of ground effects on airplanes and cars, hydrodynamics of ship-to-ship interactions, etc. [12].

Specifically, we assume $N$ slender bodies to be separated by distances comparable to their typical length, but much greater than their typical width. The near field of each separate body consists still of a state of plane strain, on which there is no direct influence of any other body. The outer field is however the sum of distributed surface tractions along all $N$ axes. By asymptotically matching, $N$ coupled integral equations are derived for the integrated normal force distribution along each axis, and these are readily solvable numerically.

Actual numerical results are presented here only for $N = 2$, for three examples involving two identical bodies, namely the cases when these bodies are (a) parallel to each other, (b) in tandem, or (c) at a right angle to each other. In the latter cases (b) and (c), the analysis enables prediction not only of the effective compliance of the foundation, but also the angle of pitching rotation induced by the interaction between the two bodies.

## 2. PRELIMINARIES

Consider a linearly-elastic, isotropic and homogeneous, half-space $y \leq 0$. Combining the equations of static equilibrium and Hooke's law, the governing equations of Navier for the displacement $\mathbf{u} = (u, v, w)$ are ([2], p. 33)

$$(1 - 2\nu)\Delta\mathbf{u} + \nabla\nabla \cdot \mathbf{u} = 0 \tag{2.1}$$

where $\nu$ is Poisson's ratio.

For the moment, we consider the response of this half-space to a load consisting of a single slender intruding body, this being a disturbance confined to a region close to a segment $-l < z < l$ of the $z$-axis. The disturbance has a characteristic length $2l$, and a characteristic width (perpendicular to the $z$-axis) of $2a \ll 2l$. Thus we can define a small parameter $\epsilon = a/l \ll 1$, and our task is to seek an approximate solution or asymptotic expansion as $\epsilon \to 0$.

This asymptotic expansion must be carried out separately in inner and outer regions, and the two expansions are then matched in a common overlap domain, as described by Van Dyke [13]. The inner region is the near neighbourhood $x$, $y = 0(a) = 0(\epsilon l)$ of the axis of loading, while the outer region is one in which all coordinates $x$, $y$, $z$ are $0(l)$.

One way to treat the inner problem is to define stretched coordinates and displacements, i.e.

$$x' = x/\epsilon,\ y' = y/\epsilon,\ z' = z$$

and

$$u' = u,\ v' = v,\ w' = w/\epsilon. \tag{2.2}$$

Then, upon substituting (2.2) into (2.1) and collecting leading-order terms in powers of $\epsilon$, we have (with $0(\epsilon^2)$ relative error)

$$(1 - 2\nu)\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right)\begin{pmatrix} u' \\ v' \end{pmatrix} + \begin{pmatrix} \partial/\partial x' \\ \partial/\partial y' \end{pmatrix}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0. \tag{2.3}$$

That is, the inner problem is, to leading order in $\epsilon$, simply one of ***plane strain***, and is a two-dimensional problem in each separate plane $z' = \text{constant}$. Once the problem for $u'$, $v'$ is solved, the axial displacement (if required) can be obtained by solving a Poisson equation

$$(1-2\nu)\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}\right)w' + \frac{\partial}{\partial z'}\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}\right) = 0 \tag{2.4}$$

for $w'$.

In Sections 3 and 4 we discuss the appropriate solutions of this purely two-dimensional problem, for some special and general cases. In doing so, we revert to the original un-scaled variables, and temporarily ignore the (parametric) dependence on the axial co-ordinate $z$. The type of plane problem of interest here is one in which there is a net force per unit (axial) length, of magnitude $F = F(z)$ at section $z$. For example, it could be that each such section is completely independent of its neighbour, as would happen if the load consisted of a linear stack of thin blocks. In such a case $F(z)$ is known in advance, and the appropriate plane-strain problem is to determine the resulting stresses and displacements. However, as is well known, the solution of such a two-dimensional problem is, by itself unsatisfactory, in that the vertical displacement $v$ tends logarithmically to infinity at a great distance from the loaded region. At the same time, while remaining bounded, the horizontal displacement does not tend to zero, as it should.

Somewhat more generally, $F(z)$ may not be known in advance. For example, if the load consists of a rigid or even an elastic beam, although the total force $\int F(z)\,dz$ may be prescribed, its distribution $F(z)$ along the beam is not known in advance, and must be treated as one of the unknowns in the problem.

To remedy these inadequacies of the inner solution, we need to examine the outer region, in which the problem remains fully three-dimensional. Since $l$ is then the appropriate length scale in *all* directions, and the loaded width is $0(a) \ll 0(l)$, the load appears to be confined to a singular line, consisting of a portion of the $z$-axis. We assume symmetry with respect to $x$, so that only the vertical force needs to be considered in the outer region.

It is convenient to employ the Papkovitch–Neuber representation ([2], p. 45)

$$2\mu\mathbf{u} = -y\nabla\psi - \nabla\phi + \kappa\psi\mathbf{j} \tag{2.5}$$

where $\mu$ is the shear modulus

$$\kappa = 3 - 4\nu \tag{2.6}$$

and $\mathbf{j}$ is the unit $y$-vector. The displacement field defined by (2.5) satisfies (1.1) if both potentials $\phi(x, y, z)$ and $\psi(x, y, z)$ satisfy Laplace's equation.

The solutions corresponding to a line of (positive-upward) vertical force, of strength $F(z)$ per unit length, are

$$\psi = \frac{1}{2\pi}\int_{-l}^{l} \frac{F(z')\,dz'}{\sqrt{(x^2+y^2+(z-z')^2)}} \tag{2.7}$$

and

$$\phi = \frac{1-2\nu}{2\pi}\int_{-l}^{l} dz' F(z') \log\left[\frac{y+\sqrt{(x^2+y^2+(z-z')^2)}}{x^2+(z-z')^2}\right]. \tag{2.8}$$

These potentials are in fact related by

$$\frac{\partial\phi}{\partial y} = (1-2\nu)\psi, \tag{2.9}$$

which ensures vanishing shear $\tau_{12}$ everywhere on the free surface $y = 0$. Note that $\phi$ is carefully chosen so as to be analytic for all $y<0$. In general, given an expression for $\psi$ such as (2.7), an expression for $\phi$ can be obtained by any indefinite integration of (2.9) with respect to $y$, but only the *particular* integral (2.8) is harmonic and gives continuous $(\partial\phi/\partial x)$ across the plane $x=0$ for $y<0$. This result is not well known and is proved in Appendix A.

On the free surface $y=0$, the displacement components corresponding to the above $\phi$, $\psi$ representations are given by

$$u(x,0,z) = -\frac{1}{2\mu}\frac{\partial\phi}{\partial x}(x,0,z)$$

$$= \frac{1-2\nu}{4\pi\mu}\, x\int_{-l}^{l}\frac{F(z')\,\mathrm{d}z'}{x^2+(z-z')^2}, \tag{2.10}$$

and

$$v(x,0,z) = \frac{1-\nu}{\mu}\,\psi(x,0,z)$$

$$= \frac{1-\nu}{2\pi\mu}\int_{-l}^{l}\frac{F(z')\,\mathrm{d}z'}{\sqrt{(x^2+(z-z')^2)}}. \tag{2.11}$$

If we now let $|x|/l\to 0$ while $|z|<l$, the asymptotic form of the integrals (2.10), (2.11) is

$$u \to \frac{1-2\nu}{4\mu}\,.\ \mathrm{sgn}\, x.\ F(z) \tag{2.12}$$

and

$$v \to \frac{1-\nu}{2\pi\mu}\left[-F(z)\log\left(\frac{x^2}{4(l^2-z^2)}\right)+\int_{-l}^{l}\frac{F(z')-F(z)}{|z'-z|}\,\mathrm{d}z'\right]. \tag{2.13}$$

The limit (2.12) is elementary; that in (2.13) is not quite so easy to derive, but is well known from similar problems in slender-body aerodynamics [6, 7, 14], and was also derived directly by Kalker [4]. A corresponding inner limit for the axial displacement $w$ can be derived, but is not needed here.

Equations (2.12) and (2.13) are the inner expansions of the outer solution, i.e. represent the singular behaviour of the line distributions of force, as we approach that line. These must match corresponding outer expansions of the inner solutions, i.e. must agree with the behaviour at a (relatively) great lateral distance from the loaded area, of the two-dimensional problems at each fixed section $z$. Such matching leads to an integral relationship between the force distribution $F(z)$ and an integrated representation $v_0(z)$ of the vertical displacement at section $z$. Either of these quantities may be considered known; if $F(z)$ is known, then $v_0(z)$ follows directly by quadrature, whereas if (as is more common) $v_0(z)$ is known, an integral equation must be solved to determine $F(z)$.

In the next two sections we set up solutions for inner problems capable of matching (2.12) and (2.13), and hence derive the integral equation, which is further discussed in Section 5.

## 3. THE INNER PROBLEM FOR A SHALLOW BODY

All plane-strain elastostatic problems in the $\zeta = x+iy$ plane can be represented ([3], p. 113) in terms of two independent analytic functions $\Omega(\zeta)$ and $\omega(\zeta)$ such that

$$2\mu(u+iv) = \kappa\Omega - \zeta\overline{\Omega'} - \bar{\omega} \tag{3.1}$$

and the stresses are given by

$$\tau_{11}+\tau_{22} = 2\Omega' + 2\overline{\Omega'} \tag{3.2}$$

$$\tau_{22} + i\tau_{12} = \Omega' + \overline{\Omega'} + \bar{\zeta}\Omega'' + \omega'. \tag{3.3}$$

Various types of boundary conditions can be applied in the finite part of the plane, and at infinity $\Omega$ and $\omega$ can grow at most logarithmically.

In the present section we consider that class of problem in which the domain of interest is the *whole* lower-half plane $y \leq 0$. A load is supposed to be applied over some prescribed segment $|x| < a$ of the axis $y = 0$, producing small displacements. Thus, this corresponds to a sharp-edged body that is not only slender but also *shallow*, in the sense that its penetration into the medium is small. However, we shall not assume that this penetration is uniform, i.e. the body need not have a flat bottom. The solutions to be presented below are of course well known, some dating from last century, but we reproduce them here, in order to examine some special featues not often considered in the classical studies.

The simplest class of such problems is the *frictionless* problem, in which the whole line $y = 0$ is free of shearing stress. In such cases, the analytic functions $\Omega$ and $\omega$ are related directly by

$$\omega = \Omega - \zeta\Omega', \tag{3.4}$$

which guarantees via (3.3) that $\tau_{12} = 0$ for all $x$ when $y = 0$. An appropriate solution for $\Omega$ is then given by (for some real constant $F$),

$$2\pi i(\zeta^2 - a^2)^{1/2}\Omega'(\zeta) = F - \frac{\mu}{1-\nu}\int_{-a}^{a} \frac{(a^2 - \xi^2)^{1/2} v'(\xi)}{\xi - \zeta}\, d\xi, \tag{3.5}$$

which makes $\Omega'$ pure imaginary on $y = 0$, $|x| > a$, so that $\tau_{22} = 0$ in that region. On the other hand, for $|x| > a$,

$$\Omega'(x - i0) = \frac{1}{2\pi}(a^2 - x^2)^{-1/2}\left[F - \frac{\mu}{1-\nu}\int_{-a}^{a} \frac{(a^2 - \xi^2)^{1/2} v'(\xi)}{\xi - x}\, d\xi\right] + \frac{1}{2}\left(\frac{i\mu}{1-\nu}\right) v'(x), \tag{3.6}$$

the integral in (3.6) being the Cauchy principal-value form.

Thus, using (3.1), $v'(x)$ is identified with the $x$-derivative of the vertical displacement under the punch $|x| < a$, while in that region the normal stress is

$$\tau_{22} = \frac{1}{\pi}(a^2 - x^2)^{-1/2}\left[F - \frac{\mu}{1-\nu}\int_{-a}^{a} \frac{(a^2 - \xi^2)^{1/2} v'(\xi)}{\xi - x}\, d\xi\right]. \tag{3.7}$$

It is immediate that

$$F = \int_{-a}^{a} \tau_{22}\, dx$$

is the net load applied by the punch. Also, we note by comparing the real parts of (3.1) and (3.3), that the $x$-derivative of the horizontal displacement under the punch is given by

$$u_x = \frac{1 - 2\nu}{2\mu}\tau_{22}, \text{ on } y = 0 \tag{3.8}$$

with $\tau_{22}$ determined by (3.7), and hence, upon integration from $x = 0$ to $x = a$,

$$u(a) = \frac{1 - 2\nu}{4\mu} F. \tag{3.9}$$

The surface displacements outside the punch, e.g. for $x > a$, satisfy

$$u_x = 0 \tag{3.10}$$

and

$$v_x = -\frac{1-\nu}{\pi\mu}(x^2-a^2)^{-1/2}F + \frac{1}{\pi}(x^2-a^2)^{-1/2}\int_{-a}^{a}\frac{(a^2-\xi^2)^{1/2}v'(\xi)}{\xi - x}\,d\xi. \tag{3.11}$$

Thus, upon integration from $x = a$,

$$\begin{aligned} u(x) &= u(a) \\ &= \frac{1-2\nu}{4\mu}F \\ &= \text{constant},\ x > a, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} v(x) = v(a) &- \frac{1-\nu}{\pi\mu}F\log\left[\frac{x+(x^2-a^2)^{1/2}}{a}\right] \\ &+ \frac{1}{\pi}\int_{-a}^{a} d\xi v'(\xi)(a^2-\xi^2)^{1/2}\int_{a}^{x}\frac{dt(t^2-a^2)^{-1/2}}{\xi - t}. \end{aligned} \tag{3.13}$$

As $x \to \infty$, (3.13) reduces to

$$v(x) \to v_0 - \frac{1-\nu}{\pi\mu}F\log\frac{2x}{a} + 0(x^{-1}), \tag{3.14}$$

where

$$v_0 = v(a) + \frac{1}{\pi}\int_{-a}^{a} d\xi v'(\xi)(a^2-\xi^2)^{1/2}\int_{a}^{\infty} dt\,\frac{(t^2-a^2)^{-1/2}}{\xi - t} \tag{3.15}$$

$$= \frac{1}{\pi}\int_{-a}^{a} dx v(x)\,(a^2-x^2)^{-1/2}. \tag{3.16}$$

The reduction of the integral (3.15)–(3.16) is a special case ($\beta = 0$) of a result proved in Appendix B. Note that if the displacement is uniform, i.e. ($v(x)$ is constant, then both (3.15) and (3.16) indicate that $v_0$ is equal to that constant value $v(x) = v(a)$.

Another important class of punch problems is that for *adhesive* contact, in which both horizontal and vertical displacements are prescribed in $|x| < a$, but $\tau_{12}$ does not necessarily vanish there. Thus (3.4) cannot be assumed to hold. It is convenient to express ($\omega(\zeta)$ in terms of an analytic continuation $\bar{\Omega}(\zeta = \Omega(\bar{\zeta})$ of $\Omega(\zeta)$ across $y = 0$, writing ([3], p. 474)

$$\omega' = -\Omega' - \bar{\Omega}' - \zeta\Omega''. \tag{3.17}$$

The appropriate solution is then given for *all* $\zeta$ by

$$2\pi i Q(\zeta)\Omega'(\zeta) = F + 2\mu\int_{-a}^{a}\frac{Q(\xi + i0)[u'(\xi) + iv'(\xi)]\,d\xi}{\xi - \zeta}. \tag{3.18}$$

The function $Q(\zeta)$ in (3.17) is defined by

$$Q(\zeta) = (\zeta + a)^{1/2 - i\beta}(\zeta - a)^{1/2 + i\beta} \tag{3.19}$$

where

$$\beta = \frac{1}{2\pi}\log\kappa, \tag{3.20}$$

and on $y = 0_{\pm}$, $|x| < a$, satisfies

$$Q(x \pm i0) = \pm i\kappa^{\mp 1/2}(a+x)^{1/2-i\beta}(a-x)^{1/2+i\beta}. \tag{3.21}$$

It may easily be verified that the stresses $\tau_{12}$ and $\tau_{22}$ corresponding to (3.18) vanish on $y = 0$, $|x| > a$ (but not in general for $|x| < a$) and that $u'(x)$ and $v'(x)$ are identified with the derivatives of the surface displacements on $y = 0_-$, $|x| < a$. Using (3.1) and (3.17), the surface displacements for $x \geqslant a$ are given by

$$u(x) + iv(x) = u(a) + iv(a) + \frac{2}{\mu}(1-\nu)\int_a^x \Omega'(t)\,\mathrm{d}t, \tag{3.22}$$

and at a great distance $x \to +\infty$ we have

$$u(x) + iv(x) = -i\frac{1-\nu}{\pi\mu}F\log\frac{2x}{a} + u_0 + iv_0 + F.(u_1 + iv_1) + 0(x^{-1}), \tag{3.23}$$

where

$$\begin{aligned} u_0 + iv_0 &= u(a) + iv(a) \\ &+ \frac{2}{\pi}(1-\nu)\kappa^{-1/2}\int_{-a}^{a}\mathrm{d}\xi[u'(\xi) + iv'(\xi)] \\ &\cdot (a+\xi)^{1/2-i\beta}(a-\xi)^{1/2+i\beta}\int_a^\infty \frac{\mathrm{d}t(t+a)^{-1/2+i\beta}(t-a)^{-1/2-i\beta}}{\xi - t} \end{aligned} \tag{3.24}$$

and

$$u_i + iv_1 = i\frac{1-\nu}{\pi\mu}\int_a^\infty \mathrm{d}t(t^2-a^2)^{-1/2}\left[1 - \left(\frac{t+a}{t-a}\right)^{i\beta}\right]. \tag{3.25}$$

It is proved in Appendix B that (3.24) simplifies to give

$$u_0 = 0, \tag{3.26}$$

and

$$\begin{aligned} v_0 &= \frac{2}{\pi}(1-\nu)\kappa^{-1/2}\int_{-a}^{a}\mathrm{d}x(a^2-x^2)^{-1/2} \\ &\cdot \left[u(x)\sin\left(\beta\log\frac{a-x}{a+x}\right) + v(x)\cos\left(\beta\log\frac{a-x}{a+x}\right)\right], \end{aligned} \tag{3.27}$$

and in Appendix C that (3.25) implies that

$$u_1 = \frac{1+2\nu}{4\mu}, \tag{3.28}$$

and

$$v_1 = \frac{1-\nu}{\pi\mu}\log\left(\frac{a_0}{a}\right) \tag{3.29}$$

where

$$\log\frac{a_0}{a} = \int_1^\infty \mathrm{d}x(1-x^2)^{-1/2}\left[1 - \cos\left(\beta\log\frac{1+x}{1-x}\right)\right]. \tag{3.30}$$

Thus, finally, from (3.22), as $x \to +\infty$,

$$u \to \frac{1-2\nu}{4\mu} F + 0(x^{-1}) \tag{3.31}$$

and

$$v \to v_0 \frac{1-\nu}{\pi\mu} F \log \frac{2x}{a_0} + 0(x^{-1}). \tag{3.32}$$

The result (3.31) is at first sight rather remarkable. In particular, the far-field limit of the *horizontal* surface displacement $u$ due to an adhesive load $F$ is *identical* to that for a frictionless load $F$. The result (3.32) for the vertical displacement $v$ is also of the same form as that for the frictionless case (3.14), but the formula (3.27) determining $v_0$ in the adhesive case differs from (3.16) in the frictionless case, and in addition we must replace the actual half-width $a$ by a parameter $a_0$ determined by (3.30).

The expression (3.30) for the effective half-width $a_0$ is a function only of $\beta$, and hence of the Poisson ratio $\nu$. This function can be computed once and for all, and $a_0/a$ is shown in Fig. 1, plotted against $\nu$. The meaning of this quantity $a_0$ is quite simple. In particular, for flat-bottomed rigid punches in which $v_0$ is just the uniform vertical displacement, the far-field properties of an adhesive punch of width $a$ are identical to that of an equivalent frictionless punch of width $a_0$. Thus, since $a_0/a > 1$, Fig. 1 indicates quantitatively the extent to which adhesion involves more of the medium in displacement. In the practical range, say $\nu = 0.35$–$0.25$, a 5–10% increase in effective contact width occurs, due to adhesion.

The expressions (3.16), (3.27) for $v_0$ have a similar interpretation as an *effective* uniform displacement. Thus, as seen in the far field, every non-flat shallow punch behaves like an equivalent flat-bottomed punch, whose uniform displacement $v_0$ is an appropriately-weighted mean displacement. For the frictionless case, only the vertical displacement $v(x)$ influences $v_0$, but in the case of adhesive contact, if there is a non-zero (but laterally anti-symmetric) horizontal displacement $u(x)$, this can also contribute to the effective far-field vertical displacement. Note that if $\beta = 0$, i.e. $\kappa = 1$ or $\nu = 1/2$, and the medium is incompressible, there is no way to distinguish frictionless and adhesive contact in the far field, since $a_0 = a$ and the formulae (3.16) and (3.27) give identical values for $v_0$.

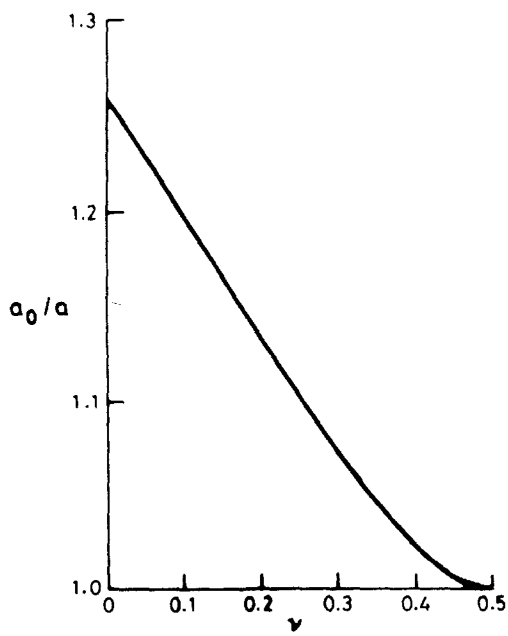


Fig. 1. Correction factor (effective width/actual width) for adhesive contact, as a function of Poisson ratio. $\nu$.

As an example of computation of the effective displacement $v_0$, consider the parabolic-bottomed adhesive punch with $u(x) = 0$ and

$$v(x) = v(0) + (v(a) - v(0))x^2/a^2. \tag{3.33}$$

Then (3.26) gives

$$v_0 = \frac{v(0) + v(a)}{2} - 2\beta^2(v(a) - v(0)). \tag{3.34}$$

Thus if $\beta = 0$ (frictionless or incompressible), then $v_0$ is simply the mean of centre and edge displacement. If $\beta > 0$, $v_0$ is closer to $v(0)$ than to $v(a)$, but the effect of adhesion on $v_0$ is quite small, since $2\beta^2 \sim 0.05$ for typical cases.

It is appropriate at this point to remark that we have not so far discussed *free* contact problems for smooth-edged curved punches, whose contact width $2a$ is *a priori* unknown, and is to be determined as part of the solution process. However, this Hertz problem is a simple extension of the present analysis, and additional condition expressing finiteness of displacements at the edge $x = a$ being used to fix $a$. In fact, Kalker's [4] original derivation was for such free problems, with $v$ given by (3.33).

If there are two or more shallow bodies in the inner region, it is not difficult to extend the above complex-function analysis to incorporate this case. However, the detailed derivations are considerably more complicated and are not pursued here. Thus, if there are two or more bodies, we assume that they are separated by a distance that is large compared to the inner length scale $a$.

## 4. A GENERAL CLASS OF INNER PROBLEMS

The two shallow-contact problems treated in Section 3 are only samples from a large range of possible inner plane-strain problems of interest. In particular, there is no difficulty in principle in solving problems involving bodies of arbitrary cross section embedded in the half-space $y \leq 0$. Such problems would occur, for example, in analysis of deep footings for long buildings.

To be more specific, let us consider a situation as sketched in Fig. 2, with a plane free surface $S_F$, consisting of that portion of the $x$-axis not occupied by the body, and let $S_B$ be the curve of contact between body and medium, lying somewhere in $y \leq 0$. We assume that the contact is adhesive, thus specifying the displacements $(u, v) = (U, V)$ everywhere on $S_B$. The problem is assumed to have lateral symmetry, so that $S_B$ is symmetric about $x = 0$, and $U$, $V$ are given odd and even functions of $x$, respectively.

The solution of every such problem can be represented in terms of the functions $\Omega(\zeta)$ and $\omega(\zeta)$ defined in eqns (3.1)–(3.3). If $F$ is the resultant upward force, then at infinity these functions are known to take the form ([3], p. 394)

$$\Omega \rightarrow -\frac{iF}{2\pi}\log\zeta + \Omega_0 + 0(\zeta^{-1}) \tag{4.1}$$

and

$$\omega \rightarrow \frac{iF}{2\pi}\log\zeta + \omega_0 + 0(\zeta^{-1}) \tag{4.2}$$

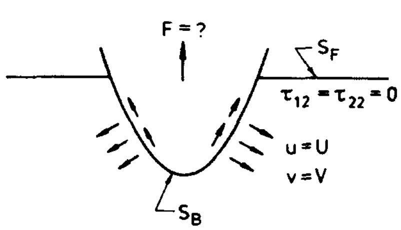


Fig. 2. General two-dimensional boundary-value problem.

where $\Omega_0$, $\omega_0$ are complex constants to be determined. In fact, as is seen by substitution into (3.1), only the particular combination $\kappa\Omega_0 - \bar{\omega}_0$ of these constants is significant, and either $\Omega_0$ or $\omega_0$ can be set to zero without loss of generality. Further, if we invoke symmetry about axis $x = 0$, we must have $u = 0$ at $x = 0$, and (3.1) then implies that

$$R(\kappa\Omega_0 - \bar{\omega}_0) = \frac{\kappa - 1}{4} F. \tag{4.3}$$

If the imaginary part of the constant $\kappa\Omega_0 - \bar{\omega}_0$ is rewritten as

$$I(\kappa\Omega_0 - \bar{\omega}_0) = 2\mu v_0 - \frac{2}{\pi}(1 - \nu) F \log\frac{2}{a_0} + \frac{F}{2\pi}, \tag{4.4}$$

then it follows by direct substitution into (3.1) that the surface displacements are given by (3.31) and (3.32), as $x \to \infty$. Our aim is to determine the unknown constants $a_0$ and $v_0$.

As illustrated by the examples in Section 3, within the strict context of the two-dimensional theory, the net force $F$ and (mean) vertical displacement $v_0$ are quite unrelated quantities, whose ultimate connection to each other can only be determined three-dimensionally. Therefore, we can split the two-dimensional boundary-value problem into two independent canonical problems, distinguished by superscripts (1) and (2), writing

$$\left.\begin{aligned} \Omega &= F\Omega^{(1)} + \Omega^{(2)} \\ \text{and} \qquad \omega &= F\omega^{(1)} + \omega^{(2)} \end{aligned}\right\}, \tag{4.5}$$

where the effective net force is *unity* for problem (1) and *zero* for problem (2).

Specifically, problem (1) corresponds to the *homogeneous* mixed boundary conditions

$$\tau_{12}^{(1)} = \tau_{22}^{(1)} = 0 \text{ on } S_F \tag{4.6}$$

and

$$u^{(1)} = v^{(1)} = 0 \text{ on } S_B, \tag{4.7}$$

with, as $x \to \infty$ on $y = 0$,

$$u^{(1)} \to \frac{1 - 2\nu}{4\mu} + 0(x^{-1}) \tag{4.8}$$

and

$$v^{(1)} \to -\frac{1 - \nu}{\pi\mu} \log\frac{2x}{a_0} + 0(x^{-1}). \tag{4.9}$$

The corresponding limiting behaviour of $\Omega^{(1)}$, $\omega^{(1)}$ at infinity is

$$\Omega^{(1)} \to -\frac{i}{2\pi}\log\zeta + \Omega_0^{(1)} + 0(\zeta^{-1}) \tag{4.10}$$

and

$$\omega^{(1)} \to -\frac{i}{2\pi}\log\zeta + \omega_0^{(1)} + 0(\zeta^{-1}) \tag{4.11}$$

where the constants $\Omega_0^{(1)}$ and $\omega_0^{(1)}$ are related by

$$\kappa\Omega_0^{(1)} - \overline{\omega_0^{(1)}} = \frac{\kappa - 1}{4} + i\left[\frac{2}{\pi}(1 - \nu)\log\frac{2}{a_0} + \frac{1}{2\pi}\right]. \tag{4.12}$$

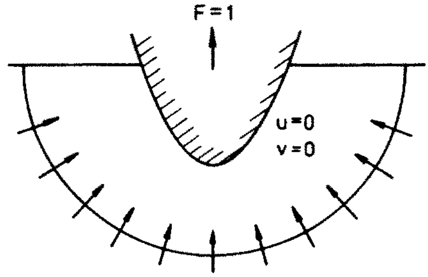


Fig. 3. Canonical problem (1), for clamped body in radial force field of unit resultant.

Physically, this problem corresponds to a half space supported by a *rigid clamp* along $S_B$, while under stress due to application of a radially-symnetric field of force, as sketched in Fig. 3. This problem is quite straightforward to solve, using numerical methods such as the hybrid finite-element and boundary-integral-equation methods described by Mei [15] in a water-wave context, or a direct boundary-integral equation method as in [16], and its solution yields the effective width parameter $a_0$ immediately.

Problem (2) still corresponds to zero-stress conditions like (4.6) on $S_F$, but to

$$u^{(2)} = U,\ v^{(2)} = V \text{ on } S_B. \tag{4.13}$$

On the free surface $y = 0$ as $x \to +\infty$, with error $0(x^{-1})$,

$$u^{(2)} \to 0 \tag{4.14}$$

and

$$v^{(2)} \to v_0. \tag{4.15}$$

The corresponding limits for $\Omega^{(2)}$, $\omega^{(2)}$ as $\zeta \to \infty$ are simply constants such that

$$\kappa\Omega^{(2)} - \overline{\omega^{(2)}} = 2\mu\, v_0 i. \tag{4.16}$$

Again, this problem has a simple physical interpretation, as sketched in Fig. 4. The contact surface $S_B$ is given the actual displacements $U$, $V$ that are prescribed by the boundary conditions, but the whole medium is then given a certain uniform vertical displacement $v_0$ at infinity, of precisely such a magnitude as to reduce to zero the resultant vertical force. This problem is also quite straightforward to solve by numerical methods like those described in [15, 16], and its solution determines the effective vertical displacement $v_0$.

We should note at once that, if the given displacements on $S_B$ are of the form $(U, V) = (0, V_0)$ where $V_0$ is constant, then the solution of this problem is trivial, namely $u^{(2)} = 0$ and $v^{(2)} = V_0$ everywhere; hence $v_0 = V_0$. That is, if the given displacement vector consists of a uniform vertical displacement only, then $v_0$ is equated with that uniform displacement. Problem (2) only requires detailed solution if the given displacement is non-uniform, or has a non-zero horizontal component.

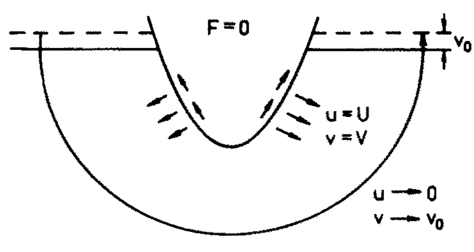


Fig. 4. Canonical problem (2), for body displaced in a medium shifted upward through $v_0$.

Some of the restrictions imposed at the beginning of the present section can be removed, in setting up numerical solutions of these two canonical problems. For example, the body curve $S_B$ could be completely submerged in the elastic half space, or non-symmetric laterally, or could consist of two separate pieces. The last extension would enable treatment of the case of two interacting bodies in each other's near field.

In summary, we have shown in the present section that eqns (3.31) and (3.32) give the correct far-field limit of the inner problem in the general case, and have indicated how the parameters $a_0$ and $v_0$ may be determined. The net force $F$ is still unknown, and we must proceed to match with the outer region in order to determine it.

## 5. THE INTEGRAL EQUATION FOR A SINGLE BODY

The process of matching together the inner expansion of the outer expansion and the outer expansion is now almost compete for a single straight slender body. All that is required is to identify the expressions (2.13) and (3.32) for the vertical displacements, since the expressions (2.12) and (3.31) for the horizontal displacements are already in exact agreement. Thus we have (restoring the explicit dependence of $F$, $v_0$ and $a_0$ upon $z$)

$$\frac{2\pi\mu}{1-\nu}\, v_0(z) = F(z)\log\left[\frac{16(l^2-z^2)}{a_0^2(z)}\right] + \int_{-l}^{l}\frac{F(z')-F(z)}{|z'-z|}\,dz'. \tag{5.1}$$

This is essentially the integral equation obtained independently by Kalker [4] and Sivashinsky [9].

Note however, that the result (5.1) applies in situations considerably more general than those described by previous authors. In particular, Sivashinsky's derivation in [9] was strictly limited to shallow frictionless punches.† Although Kalker [4] formulated some more general cases, his integral equation equivalent to (5.1) was also only derived either for the shallow frictionless case or for the incompressible case $\nu = 1/2$ when, as we have seen, adhesive and frictionless punches are indistinguishable in the far field.

In contrast, the present derivation reveals that (5.1) applies to any type of load applied over a geometrically slender surface of contact, providing we give an appropriate interpretation to the effective width $2a_0(z)$ and effective vertical displacement $v_0(z)$ at each separate section $z = \text{constant}$. Such an interpretation may require a prior numerical solution of a plane-strain problem as discussed in Section 4, or may be via an explicit formula for the shallow-contact problems discussed in Section 3. In any case, $2a_0$ and $v_0$ have physical and intuitive interpretations, as the width and vertical displacement of an equivalent flat frictionless punch.

If the force distribution $F(z)$ is *given*, eqn (5.1) is an explicit formula for the resulting displacement $v_0(z)$. This situation is not without practical significance, since it corresponds to a body with a perfectly flexible backbone, as could be achieved to a close approximation by an elongated stack of closely-spaced, but non-touching bricks. In such a case, the force $F(z)$ at any section $z$ may be equated with the known weight of the brick located at that section, and (5.1) then determines that brick's vertical displacement, in a manner that is influenced by the reactions of the elastic medium to the other bricks in the stack.

In particular, if the given force distribution is *uniform*, i.e. $F$ is constant, then the integral in (5.1) vanishes and

$$\frac{2\pi\mu v_0(z)}{1-\nu} = F\log\left[\frac{16(l^2-z^2)}{a_0^2(z)}\right]. \tag{5.2}$$

An important further specialization is to bodies with an elliptical (effective) width function

$$a_0(z) = \epsilon l(1 - z^2/l^2)^{1/2} \tag{5.3}$$

for which $v_0$ is also constant, with

$$\frac{2\pi\mu v_0}{1-\nu} = F\log(16/\epsilon^2). \tag{5.4}$$

†And appears to lack the important factor of 16 in the logarithm.

The result (5.4) is the correct small-$\epsilon$ limit of the known exact solution for an elliptical frictionless flat punch, with errors significantly less than 10% for $\epsilon$ values near 0.1 (see [4]).

A generalization of (5.3) is to plan-forms of the type

$$a_0(z) = \epsilon l(1 - z^2/l^2)^\alpha, \tag{5.5}$$

for which, with $F = \text{constant}$,

$$\frac{2\pi\mu}{1-\nu} v_0(z) = F\left[\log\frac{16}{\epsilon^2} + (1-2\alpha)\log(1 - z^2/l^2)\right] \tag{5.6}$$

We note that as $|z| \to l$,

$$v_0 \to \pm\infty \text{ if } \alpha \gtrless 1/2. \tag{5.7}$$

That is, if we attempt to support a uniform load $F$ on a footing that is more pointed ($\alpha > 1/2$) than an ellipse, the ends will be displaced in the same direction as the load, but by much more than the middle sections. Conversely, a blunt ($\alpha < 1/2$) plan form, when uniformly loaded, reacts *against* the load at its ends. Only the elliptic case $\alpha = 1/2$ produces a finite end-displacement, for uniform loads. This type of end singularity is not unfamiliar in slender-body theory [7], but is quite weak, the infinities being logarithmic. Figure 5 shows displacements corresponding to (5.6) at $\epsilon = 0.2$ for various values of $\alpha$, indicating how localised to the ends is the singularity.

Prescribed non-uniform loads $F(z) \neq \text{const.}$ may be analysed most easily by assuming that $F(z)$ is of polynomial form, since a fundamental property [7] of the integral in (5.1) is that polynomial $F(z)$ gives polynomial values for the integral. For example, a quadratic loading that vanishes at the ends, namely

$$F(z) = F(0)(1 - z^2/l^2) \tag{5.8}$$

leads to

$$\frac{2\pi\mu}{1-\nu}\frac{v_0(z)}{F(0)} = (1 - z^2/l^2)\log\left[\frac{16(l^2 - z^2)}{a_0^2(z)}\right] + 3z^2/l^2 - 1. \tag{5.9}$$

This result (5.9) is displayed in Fig. 6 for rectangular sections $a_0 = \epsilon l = \text{constant}$, for various slenderness ratios $\epsilon$. There is now no end singularity, and $v_0(z)$ is everywhere bounded and

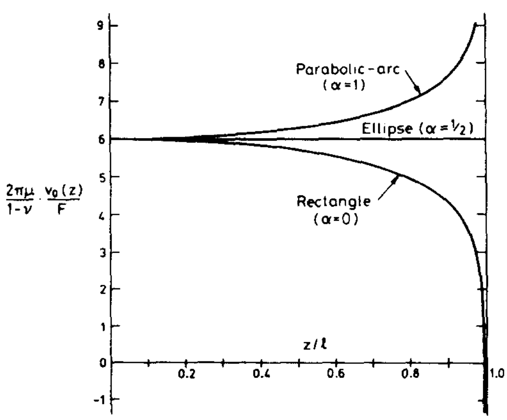


Fig. 5. Displacement $v_0(z)$ produced by a uniform load $F = \text{constant}$, for various plan forms of slenderness $\epsilon = 0.2$.

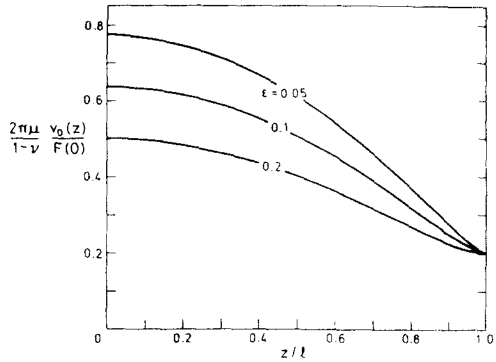


Fig. 6. Displacement $v_0(z)$ produced by a quadratic load $F(z) = F(0)(1 - z^2/l^2)$, for a rectangular plan-form of various slendernesses $\epsilon$.

positive, irrespective of the plan form $a_0(z)$. The ratio between centre and end deflection is

$$v_0(0)/v_0(l) = \log\left(\frac{4}{\epsilon}\right) - 1/2, \tag{5.10}$$

if $\epsilon = a_0(0)/l$; thus the end deflection is generally small compared to the centre deflection, for the loading (5.8).

We now turn to the more-practical problem, in which $F(z)$ is unknown *a priori* and (5.1) is an integral equation to determine it. The most straightforward case to analyse is that of a *rigid* body, in which $v_0(z)$ is prescribed. However, it is interesting to note in passing that (5.1) may also be used for an *elastic* beam, in which $v_0(z)$ is not prescribed directly, but in which $v_0(z)$ and $F(z)$ are further related by a suitable differential equation. The result is a system of coupled integral and differential equations for the pair of unknowns $F(z)$ and $v_0(z)$. Solution of this problem is left for the future.

Numerical solutions of (5.1) for $v_0(z) = \text{const.}$, were obtained by Panek and Kalker [8] using the polynomial property of the integral, referred to above. Their results used 30 polynomial coefficients, whose values were determined by forcing the quation to hold at 30 distinct values of $z$. Panek and Kalker attempted to solve for a rectangular base $a_0 = \text{constant}$, but, for reasons related to the end effect singularity discussed above, were forced to round off a small section near the ends. This rounding is not resolvable with as few as 30 polynomial terms, and hence apparently-converged results were obtained. However, there was some indication of apparent numerical instability, evidenced by waviness in the output $F(z)$.

We use a simple direct numerical method, evaluating the integral in (5.1) by the mid-point rule. Thus we write

$$\int_{-l}^{l} \frac{F(z') - F(z_i)}{|z' - z_i|}\, dz' \simeq \sum_{j=1}^{N} A_{ij} F(z_j), \tag{5.11}$$

where $\{z_j\}$ is a set of points in $(-l, l)$, and the matrix $A_{ij}$ is given by

$$A_{ij} = \frac{\Delta z_j}{|z_i - z_j|},\ j \neq i, \tag{5.12}$$

and

$$A_{ii} = -\sum_{\substack{j\neq i\\ j=1}}^{N} A_{ij}, \tag{5.13}$$

where $\Delta z_j$ is the length of the interval whose mid-point is $z_j$. Thus (5.1) is reduced to a set of $N$

linear equations in $N$ unknowns $F(z_j)$, namely

$$\frac{2\pi\mu}{1-\nu}\,v_0(z_i)=\sum_{j=1}^{N}\left[A_{ij}+\delta_{ij}\log\frac{16(l^2-z_i^2)}{a_0^2(z_i)}\right]F(z_j), \tag{5.14}$$

which can be solved by any convenient algorithm.

The above numerical method has the usual accuracy of the mid-point rule, i.e. its errors decay like $N^{-2}$ and thus tend to be smaller than the 3rd decimal place for $N>30$. The programme gives exact answers for elliptic sections, and reproduces (5.8), when values of $v_0$ computed from (5.9) (i.e. as plotted in Fig. 6) are inputted. However, it *fails completely* (as presumably does the numerical method of [8], to which it is formally almost equivalent) for exactly-rectangular plan forms $a_0(z)=$ constant, with $v_0=$ constant as input. More generally, if a family of plan forms such as (5.5) is tried, the method converges rapidly for all ends with $\alpha \geqslant 1/2$, but never converges for $\alpha<1/2$. It is indeed not clear that the integral equation (5.1) even possesses a solution for $v_0=$ const. in such cases.

Clearly, a fully three-dimensional and essentially numerical theory is needed for extremely blunt-ended plan forms. However, approximate results of sufficient accuracy may be obtained from the present theory, by suitably smoothing the ends. For example, the system (5.14) converges well with $v_0=$ const. and

$$a_0(lt)=\epsilon l(1-t^2)^{1/2}\left[1+\frac{1}{2}t^2+\frac{3}{8}t^4+\frac{5}{16}t^5+\frac{35}{128}t^8\right] \tag{5.15}$$

which defines a nearly-rectangular plan form, having 95% of the area of the circumscribing rectangle. The computed compliance factor $\gamma$, defined by

$$\gamma=\frac{2l\mu}{1-\nu}\,v_0\Big/\int_{-l}^{l}F(z)\,\mathrm{d}z \tag{5.16}$$

is plotted in Fig. 7 as the dashed curve, against the aspect ratio

$$\epsilon_A=\int_{-l}^{l}a_0(z)\,\mathrm{d}z/(2l^2) \tag{5.17}$$

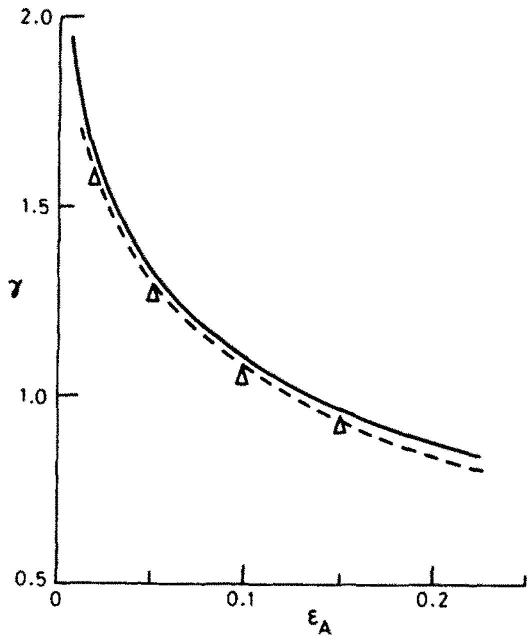


Fig. 7. Compliance factor $\gamma$ for ellipses (solid line), near-rectangles (dashed line), and as computed in Ref. [8] (triangles), as a function of aspect ratio $\epsilon_A=$ area/length$^2$ of plan form.

which takes the value 0.95 $\epsilon$ in the case of the plan form (5.15). For comparison, the compliance of an elliptical body is

$$\gamma = \frac{1}{2\pi} \log (16/\epsilon^2) \tag{5.18}$$

where $\epsilon_A = (\pi/4)\ \epsilon$, and this is shown as the solid line in Fig. 7. The results reported in [8] are also shown, by triangular points. All results are close, suggesting an almost-universal curve for $\gamma = \gamma(\epsilon_A)$.

## 6. INTERACTIONS BETWEEN BODIES

Consider $N$ slender bodies, where two typical bodies (the $j$'th and $k$'th) are as sketched in Fig. 8. These bodies are situated at distances from each other of the order of their typical lengths.

First of all, we shall show that for separations of this magnitude the lateral translation and the rotation about the longitudinal axis of one body is hardly affected by other bodies. Thus only the vertical displacement and vertical force are essential as in the case of an isolated body treated so far. Using (5.4) to estimate the typical vertical displacement, the rotation of a body due to its own loading can be estimated as,

$$0(v_0/a_0) = 0\left(\frac{1-\nu}{2\pi\mu}\frac{F}{a_0}\log(16/\epsilon^2)\right).$$

The induced rotation due to the vertical displacement of an adjacent body is, from (2.11),

$$0(v_x) = 0\left(\frac{1-\nu}{2\pi\mu}\frac{F}{l}\right).$$

Thus the latter rotation is $0(\epsilon \log (16/\epsilon^2)$ times that of the former and is negligible. On the other hand, the induced horizontal displacement due to interaction is $0[(1-2\nu/4\mu)F]$; the corresponding $u_x(x, 0, z)$ is $0[(1-2\nu/4\mu)(F/l)]$ and is equally negligible in determining the total stress field as can be seen from (3.18).

Thus the outer solution is obtained by distributing vertical pressure along the centre line of each body, which is assumed to be a straight line segment $L_k$, $k = 1, 2, \ldots, N$ lying in the plane $y = 0$. Denoting the field point by $\mathbf{r} = (x, y, z)$ and a point on the axis $L_k$ by $\mathbf{r}_k = (x_k, 0, z_k)$, the Papkovitch–Neuber $\psi$-potential can be generalized from (2.7) to

$$2\pi\psi(\mathbf{r}) = \sum_{k=1}^{N} \int_{L_k} \frac{F_k(\mathbf{r}_k)}{|\mathbf{r}-\mathbf{r}_k|}\, d\mathbf{r}_k \cdot \mathbf{e}_k \tag{6.1}$$

where $F_k(\mathbf{r}_k)$ is the upward vertical force per unit length of $L_k$, and $\mathbf{e}_k$ is a unit vector directed along $L_k$. That is, if $\mathbf{b}_k$, $\mathbf{c}_k$ are end-points of $L_k$,

$$\mathbf{e}_k = (\mathbf{c}_k - \mathbf{b}_k)/|\mathbf{c}_k\mathbf{b}_k|. \tag{6.2}$$

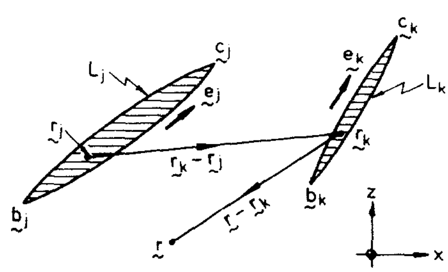


Fig. 8. Sketch of geometry for general $N$-body problem, showing two such bodies.

Now we allow the field point $\mathbf{r}$ to approach $L_k$. Define $\rho_k$ as the perpendicular distance from $\mathbf{r}$ to the segment $L_k$, i.e.

$$\rho_k = |\mathbf{e}_k \times (\mathbf{r} - \mathbf{b}_k)|. \tag{6.3}$$

Then, by analogy with (2.13), as $\rho_k \to 0$,

$$\begin{aligned} 2\pi\psi(\mathbf{r}_k) \to & - F_k(\mathbf{r}_k) \log \left[\frac{\rho_k^2}{4(\mathbf{r}_k - \mathbf{b}_k)\cdot(\mathbf{c}_k - \mathbf{r}_k)}\right] \\ & + \int_{L_k} \frac{F_k(\mathbf{r}'_k) - F_k(\mathbf{r}_k)}{|\mathbf{r}'_k - \mathbf{r}_k|} \mathrm{d}\mathbf{r}'_k \cdot \mathbf{e}_k \\ & + \sum_{\substack{j=1 \\ (j\neq k)}}^{N} \int_{L_j} \frac{F_j(\mathbf{r}_j)}{|\mathbf{r}_k - \mathbf{r}_j|} \mathrm{d}\mathbf{r}_j \cdot \mathbf{e}_j. \end{aligned} \tag{6.4}$$

In particular, if we set $y = 0$ and use (2.5), the r.h.s. of (6.4) is $(2\pi\mu/(1-\nu)) \cdot v_k$, where $v_k$ is the vertical displacement of the $k$th body, Thus, upon matching with (3.32), (in which $\rho_k$ plays the role of the lateral coordinate $x$), we have

$$\begin{aligned} \frac{2\pi\mu}{1-\nu} v_k(\mathbf{r}_k) = & F_k(\mathbf{r}_k) \log \left[\frac{16(\mathbf{r}_k - \mathbf{b}_k)\cdot(\mathbf{c}_k - \mathbf{r}_k)}{a_k^2(\mathbf{r}_k)}\right] \\ & + \int_{L_k} \frac{F_k(\mathbf{r}'_k) - F_k(\mathbf{r}_k)}{|\mathbf{r}'_k - \mathbf{r}_k|} \mathrm{d}\mathbf{r}'_k \cdot \mathbf{e}_k \\ & + \sum_{\substack{j=1 \\ (j\neq k)}}^{N} \int_{L_j} \frac{F_j(\mathbf{r}_j)}{|\mathbf{r}_k - \mathbf{r}_j|} \mathrm{d}\mathbf{r}_j \cdot \mathbf{e}_j. \end{aligned} \tag{6.5}$$

All vectors $\mathbf{r}_j$, $\mathbf{r}_k$, $\mathbf{r}'_k$ lie now in the plane $y = 0$.

Equation (6.5) is the natural generalization to $N$ bodies, of the integral equation (5.1). It is a relationship connecting the $N$ force functions $(F_k(\mathbf{r}_k))$ with the $N$ effective displacements $v_k(\mathbf{r}_k)$, see $v_0(z)$ for a single body. It involves the effective half-width function $a_k(\mathbf{r}_k)$ of the $k$th body, see $a_0(z)$ for a single body. Apart from a translation into vector language, the only new feature of (6.5), relative to (5.1), is the last term, which involves a summation at the $k$th body, of contributions to the displacement induced by the force fields $F_j$ of all *other* bodies $j \neq k$. Hence the complete set of $N$ equations (6.5), for $k = 1, 2, \ldots, N$, constitutes a set of $N$ coupled integral equations in $N$ unknown functions $F_k(\mathbf{r}_k)$.

For example, suppose all bodies are longitudinally rigid. Then the vertical displacement can be resolved into a uniform upward translation (heave) $\alpha_k$, and a rotation (pitch) $\beta_k$ about the mid-point $\mathbf{r}_k^0 = (\mathbf{b}_k + \mathbf{c}_k)/2$; thus

$$v_k(\mathbf{r}_k) = \alpha_k + \beta_k(\mathbf{r}_k - \mathbf{r}_k^0)\cdot\mathbf{e}_k. \tag{6.6}$$

The force distribution $F_k$ can be similarly decomposed, into

$$F_k = \alpha_k F_k^\alpha + \beta_k F_k^\beta. \tag{6.7}$$

Now $F_k^\alpha$ satisfies (6.5) with $v_k = 1$, while $F_k^\beta$ satisfies (6.5) with $v_k = (\mathbf{r}_k - \mathbf{r}_k^0)\cdot\mathbf{e}_k$. Once these integral equations are solved for $F_k^\alpha$, $F_k^\beta$, the heave and pitch displacements $\alpha_k$, $\beta_k$ are determined from the equilibrium conditions for each body, namely

$$\int_{L_k} \mathrm{d}\mathbf{r}_k \cdot \mathbf{e}_k[\alpha_k F_k^\alpha(\mathbf{r}_k) + \beta_k F_k^\beta(\mathbf{r}_k)] = -W_k \tag{6.8}$$

$$\int_{L_k} \mathrm{d}\mathbf{r}_k \cdot \mathbf{e}_k(\mathbf{r}_k - \mathbf{r}_k^0) \cdot \mathbf{e}_k[\alpha_R F_k^\alpha(\mathbf{r}_k) + \beta_k F_k^\beta(\mathbf{r}_k)] = -W_k \bar{l}_k \tag{6.9}$$

where $W_k$ is the weight of the $k$th body and $\bar{l}_k$ is the distance of its centre of gravity from its mid-pont.

The above is a very general framework, within which multiple-footing problems in foundation engineering can be solved. To illustrate its use, we consider here two identical bodies, symmetrically located with respect to a horizontal axis between them, as in Fig. 9. If we concentrate on one particular body, $L_1$, then it is convenient to take that body to be in the $z$-axis, from $z = -l$ to $z = +l$, in wich case (6.5) reduces to (with $F = F_1$, $v_0 = v_1$, $a_0 = a_1$)

$$\frac{2\pi\mu}{1-\nu} v_0(z) = F(z) \log \frac{16(l^2 - z^2)}{a_0^2(z)}$$

$$+ \int_{-l}^{l} \frac{F(z') - F(z)}{|z' - z|} \mathrm{d}z' + \int_{L_2} \frac{F_2(t)}{R} \mathrm{d}t, \tag{6.10}$$

where $t$ measures arc-length along the other body $L_2$ and $R = R(t, z)$ is the distance from the point on $L_2$ parametrized by $t$, to the point $z$ on $L_1$. In general, we need to write down a similar equation for $L_2$, and the result is a set of two coupled integral equations for $F(z)$, $F_2(t)$.

However, if the bodies $L_1$ and $L_2$ are identical, both with respect to geometry and loading, then $F_2 = F_1 = F(z)$ and $v_2 = v_1 = v_0(z)$ satisfy a single integral equation (6.10), which is identical to (5.1) except for the last term. Suppose that, as sketched in Fig. 9, body 2 is placed at an angle $\theta$ to body 1, with centres and distance $h$ apart. Then the distance $R(t, z)$ to use in (6.10) is given by

$$R^2 = z^2 + t^2 - 2zt \cos\theta + h^2 + 2h(z+t)\sin(\theta/2). \tag{6.11}$$

In particular, if $\theta = 0$, the bodies are parallel to each other at a distance $h$ apart, and

$$R^2 = h^2 + (z - t)^2. \tag{6.12}$$

Figure 10 shows computed results from (6.10) for $F(z)$, in the case of two parallel ($\theta = 0$) identical elliptical rigid footings $v_0 = \text{const.}$, for various values of $h$, at slenderness $\epsilon = 0.1$. Although these results are for ellipses, the remarks in Section 5 suggest that they are also applicable to other shapes. This was checked directly in the present case by repeating the computations for the nearly-rectangular plan-form (5.15), with results within 2% of those of the ellipse with the same aspect ratio.

The effect of reduction of $h$ from the value $h = \infty$ at which the single-body results hold, is to reduce the load required to produce a given displacement, and to concentrte more of it near the

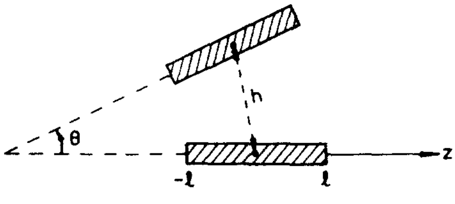


Fig. 9.

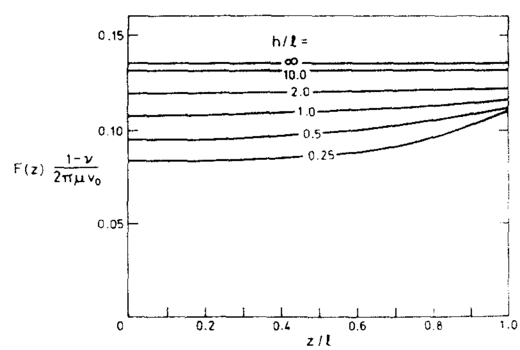


Fig. 10.

Fig. 9. Special case of two bodies at an angle $\theta$ and mid-point separation $h$.

Fig. 10. Load distribution $F(z)$ on one of two parallel ($\theta = 0$) bodies with an elliptic plan form of slenderness $\epsilon = 0.1$, for various separations $h$.

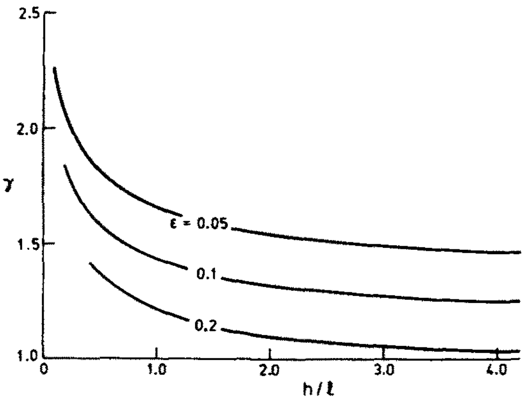


Fig. 11. Compliance factor $\gamma$ of one of two parallel elliptic bodies, as a function of separation $h$, for various slendernesses $\epsilon$.

ends. Figure 11 shows the compliance factor $\gamma$ defined by Equation (5.16), for one of the bodies, as a function of $h$, for various slendernesses $\epsilon$. As $h$ decreases, the apparent compliance of each separate body increases, since it is being displaced by the other body as well as by itself. The model ceases to have validity for small $h$, and certainly must fail for $h < 2\epsilon l$, when the two bodies formally touch each other. However, the results seem reasonable even at $h = 2\epsilon l$, where the predicted compliance is approximately the same as that of a single body of twice the width.

In the above case of identical parallel bodies, there is no pitching moment, i.e. $\beta = 0$. At any non-zero value of $\theta$, we must take pitching into account, and hence solve the integral equation (6.10) twice, as suggested by eqn (6.7); once for $F^\alpha$ with $v_0 = 1$, and once for $F^\beta$ with $v_0 = z$. Then if, for simplicity, we assume that each body has weight $W$ and is symmetric about its mid-point, the heave $\alpha$ and pitch $\beta$ are determined from

$$\alpha \int_{-l}^{l} F^\alpha \,dz + \beta \int_{-l}^{l} F^\beta \,dz = -W \tag{6.13}$$

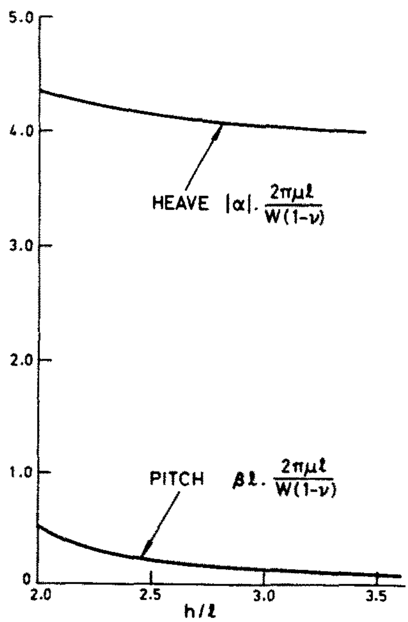


Fig. 12. Heave $\alpha$ and pitch $\beta$ of one of two tandem ($\theta = \pi$) bodies with an elliptic plan form of slenderness $\epsilon = 0.1$, as a function of (mid-point) separation $h$.

and

$$\alpha \int_{-l}^{l} zF^{\alpha}\, dz + \beta \int_{-l}^{l} zF^{\beta}\, dz = 0. \tag{6.14}$$

Figure 12 shows $\alpha$ and $\beta$ for *tandem* ($\theta = \pi$) elliptical ($\varepsilon = 0.1$) footings as functions of $h$. With $W > 0$, we find that $\alpha < 0$ and $\beta > 0$, indicating that the closest points are displaced the most, the configuration adopting a V-shape. If the computations are repeated for various slendernesses $\epsilon$, the pitch angle $\beta$ remains essentially unchanged, while the heave $|\alpha|$ varies with $\epsilon$ in a manner essentially given by Fig. 7.

Naturally, the maximum interaction occurs when the bodies nearly touch. i.e. when $h = 2l$, in the tandem case. In principle, the theory should cease to be valid then, but the results seem to be reasonable up to quite close to $h = 2l$. Figure 12 indicates that the effect of pitch is always small compared to heave, with $\beta l < 0.17|\alpha|$; for example, the difference between the end displacements is never more than 25% of the maximum displacement.

Figure 13 shows corresponding computations at $\theta = 90°$, i.e. for two identical bodies at right angles. In addition to cases where the bodies are separated ($h/l > \sqrt{2}$), we also include the results for L-shaped ($h/l = \sqrt{2}$) and cruciform ($h/l < \sqrt{2}$) configurations, with the assumption that the two bodies impose no direct constraint on each other at the point where they touch. The symmetric cross with $h/l = 0$ gives the largest vertical displacement, while maximum pitch occurs at $h/l \simeq 1$, i.e. when the short-arm to long-arm ratio is about 0.17. These global results seem to be reasonable, despite the original assumption that the bodies must be well separated at all points. However, the local response (stresses or displacements) near the intersection cannot be expected to be accurately predicted by the present theory.

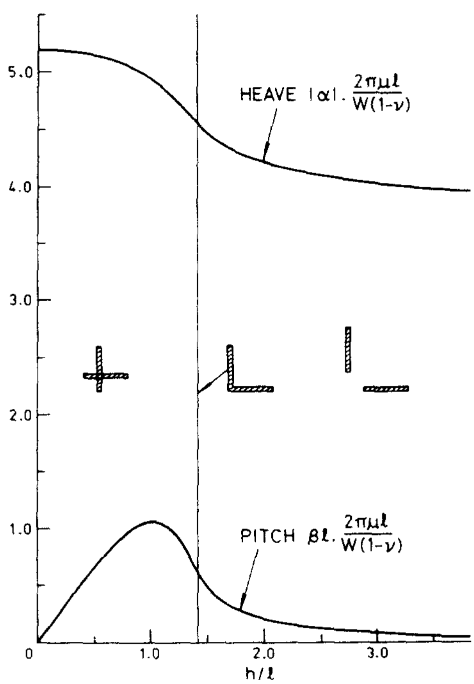


Fig. 13. Heave $\alpha$ and pitch $\beta$ of one of two bodies at a right angle, with an elliptic plan form of slenderness $\epsilon = 0.1$, as a function of (mid-point) separation $h$.

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## APPENDIX A

Equation (2.8) may be written as

$$\phi = \frac{1-2\nu}{2\pi}\int_{-l}^{l} dz'\, F(z')\,[\log(y+\sqrt{(x^2+y^2+(z-z')^2)}) - 2\log\sqrt{(x^2+(z-z')^2)}]. \tag{A1}$$

It is easy to check that each logarithmic term in the integrand is harmonic. The expression (A1) satisfies (2.9) for any choice of the constant multiplying the second logarithm; our aim is to show that only the choice "$-2$" is correct. Differentiating (A1) with respect to $x$, we obtain

$$\frac{\partial\phi}{\partial x} = \frac{1-2\nu}{2\pi}\int_{-l}^{l} dz'\, F(z')\left[\frac{x}{(y+R)R} - \frac{2x}{x^2+(z-z')^2}\right] \tag{A2}$$

where $R^2 = x^2+y^2+(z-z')^2$. If we let $x$ tend to zero in the integrand of (A2), then a contribution, if any, to the integral can occur only in the immediate neighbourhood of $z' = z$. This applies not only to the second term, which is well-known to behave as a $\delta$-function, but also to the first term for all $y<0$, since when $x=0$ and $z'=z$, the denominator vanishes since $R \to |y|$. Thus,

$$\phi_x(0_+, y, z) = \frac{1-2\nu}{2\pi}F(z)\lim_{x\downarrow 0} x\int_{-\infty}^{\infty}\left[\frac{1}{y+\sqrt{(x^2+y^2+Z^2)}}\,\frac{1}{\sqrt{(x^2+y^2+Z^2)}} - \frac{2}{x^2+Z^2}\right]dZ \tag{A3}$$

$$= \frac{1-2\nu}{2\pi}F(z)\left[\lim_{x\downarrow 0}\frac{2x}{r}\int_0^{\infty}\frac{d\lambda}{\cosh\lambda + y/r} - 2\right] \tag{A4}$$

after the substitution $Z = r\sinh\lambda$ in the first integral of (A3), where $r^2 = x^2+y^2$. The integral with respect to $\lambda$ is elementary, and takes the value $(1-y^2/r^2)^{-1/2} = r/x$. Hence, if and only if the constant multiplying the second logarithm in (A1) is "$-2$", we have

$$\phi_x(0_+, y, z) = 0, \quad y<0, \tag{A5}$$

and a similar result applies to $\phi_x(0_-, y, z)$. Thus, this expression for $\phi$ is continuous across the plane $x=0$, for all $y<0$. Note that this continuity is not maintained across the singular line at $y=0$; if $y=0$, then clearly

$$\phi_x(0_\pm, 0, z) = \pm\frac{1-2\nu}{2\pi}F(z), \tag{A6}$$

a result that is important in matching the horizontal displacements.

## APPENDIX B

Define

$$K(x) = \frac{2}{\pi i}(1-\nu)Q(x+i0)\int_a^{\infty}\frac{dt}{Q(t)(t-x)}, \tag{B1}$$

where $Q(\zeta)$ is as in (3.19) and $|x|<a$. We first show that

$$K(-a) = 0 \tag{B2}$$

and

$$K(a) = 1. \tag{B3}$$

The result (B2) is obvious from the definition of $Q$, since $Q(-a) = 0$, and the integral in (B1) is bounded at $x = -a$. The result (B3) requires consideration of the behaviour as $x \to a_-$ of the integral. Thus, setting $x = a - \epsilon$, $t = a + \epsilon\tau$, we have

$$K(a) = \frac{2}{\pi}(1-\nu)\kappa^{-1/2}(2a)^{1/2-i\beta} \lim_{\epsilon \to 0_+} \epsilon^{1/2+i\beta}$$

$$\int_0^\infty \frac{d\tau}{\tau+1}(2a+\epsilon\tau)^{-1/2+i\beta}(\epsilon\tau)^{-1/2-i\beta}$$

$$= \frac{2}{\pi}(1-\nu)\kappa^{-1/2}\int_0^\infty \frac{d\tau}{\tau+1}\tau^{-1/2-i\beta}. \tag{B4}$$

The integral in (B4) can be evaluated as a Beta function, and the result (B3) follows.

Now eqn (3.24) states that

$$u_0 + iv_0 = u(a) + iv(a) - \int_{-a}^{a} K(\xi)(u'(\xi) + iv'(\xi))\, d\xi$$

$$= u(a) + iv(a) - [K(\xi)(u(\xi) + iv(\xi))]_{-a}^{a}$$

$$+ \int_{-a}^{a} K'(\xi)[u(\xi) + iv(\xi)]\, d\xi$$

i.e.

$$u_0 + iv_0 = \int_{-a}^{a} K'(x)[u(x) + iv(x)]\, dx, \tag{B5}$$

upon use of (B2) and (B3).

We must therefore evaluate the derivative $K'(x)$. Now using the result

$$(\zeta^2 - a^2)Q'(\zeta)/Q(\zeta) = \zeta + 2i\beta a, \tag{B6}$$

that follows by a logarithmic derivative of (3.19), we have

$$K'(x) = \frac{2}{\pi i}(1-\nu)\frac{Q(x+i0)}{x^2-a^2}I \tag{B7}$$

where

$$I = (x + 2i\beta a)\int_a^\infty \frac{dt}{Q(t)(t-x)} + (x^2 - a^2)\int_a^\infty \frac{dt}{Q(t)(t-x)^2}.$$

By direct differentiation, using $Q(t)\cdot\overline{Q(t)} = t^2 - a^2$, we find that

$$\frac{d}{dt}\left[-\frac{\overline{Q(t)}}{t-x}\right] = \frac{(x+2i\beta a)}{Q(t)(t-x)} + \frac{(x^2-a^2)}{Q(t)(t-x)^2}. \tag{B8}$$

Hence

$$I = \left[-\frac{\overline{Q(t)}}{t-x}\right]_a^\infty = -1. \tag{B9}$$

Thus

$$K'(x) = \frac{2}{\pi i}(1-\nu)\frac{Q(x+i0)}{a^2-x^2}$$

$$= \frac{2}{\pi}(1-\nu)\kappa^{-1/2}(a+x)^{-1/2-i\beta}(a-x)^{-1/2+i\beta} \tag{B10}$$

and hence (B5) implies

$$u_0 + iv_0 = \frac{2}{\pi}(1-\nu)\kappa^{-1/2}\int_{-a}^{a} dx(a^2-x^2)^{-1/2}\left(\frac{a-x}{a+x}\right)^{i\beta}(u(x)+iv(x)). \tag{B11}$$

Taking real and imaginary parts of (B11) gives

$$u_0 = \frac{2}{\pi}(1-\nu)\kappa^{-1/2}\int_{-a}^{a} dx(a^2-x^2)^{-1/2}[u(x)\cos\lambda(x) - v(x)\sin\lambda(x)] \tag{B12}$$

and

$$v_0 = \frac{2}{\pi}(1-\nu)\kappa^{-1/2}\int_{-a}^{a} dx(a^2-x^2)^{-1/2}[u(x)\sin\lambda(x) + v(x)\cos\lambda(x)], \tag{B13}$$

where

$$\lambda(x) = \beta \log\frac{a-x}{a+x}.$$

But since $u$ is odd, $v$ even, and $\lambda$ odd in $x$, the integral in (B12) vanishes to give (3.26), while that in (B13) yields (3.27). Setting $\beta = 0$ gives the result (3.16).

## APPENDIX C

Equation (3.25) states that

$$u_1 = \frac{1-\nu}{\pi\mu}\int_a^\infty dt(t^2-a^2)^{-1/2}\sin\left[\beta\log\left(\frac{t+a}{t-a}\right)\right] \tag{C1}$$

and

$$v_1 = \frac{1-\nu}{\pi\mu}\int_a^\infty dt(t^2-a^2)^{-1/2}\left\{1-\cos\left[\beta\log\left(\frac{t+a}{t-a}\right)\right]\right\}. \tag{C2}$$

The result (C2) for $v_1$ yields the formulae (3.29), (3.30) of the text immediately, upon use of the change of variable $x = t/a$. Hence our task is to show that $u_1 = (1-2\nu)/4\mu$.

We first observe that

$$\int_{-\infty}^{\infty}\frac{d\zeta}{Q(\zeta)} = -\pi i \tag{C3}$$

where $Q(\zeta)$ is as given by (3.19) and the path of integration passes just above the real axis. If we complete a closed path with a semi-circle at infinity in the upper half plane, then since $Q(\zeta)\to\zeta + O(\zeta^{-1})$ as $|\zeta|\to\infty$, the integral along the semi-circle takes the value $\pi i$. Thus, since $Q(\zeta)$ is analytic, and the closed-path integral vanishes by Cauchy's theorem, (C3) is true. Thus

$$\left(\int_{-\infty}^{-a} + \int_a^\infty\right)\frac{d\zeta}{Q(\zeta)} = -\pi i - \int_{-a}^{a}\frac{dx}{Q(x+i0)}. \tag{C4}$$

The integral on the right of (C4) is over a path just above the branch cut between $-a$ and $a$. In fact integrals along any closed curve surrounding that cut must take the value $2\pi i$, again by noting that $Q\to\zeta$ at infinity. Thus

$$-\int_{-a}^{a}\frac{dx}{Q(x+i0)} + \int_{-a}^{a}\frac{dx}{Q(x-i0)} = 2\pi i. \tag{C5}$$

But since $Q(x-i0) = -\kappa Q(x+i0)$ by (3.21), this means that

$$\int_{-a}^{a}\frac{dx}{Q(x+i0)} = -\frac{2\pi i\kappa}{\kappa+1}. \tag{C6}$$

Thus (C4) states that

$$\left(\int_{-\infty}^{-a} + \int_a^\infty\right)dt(t^2-a^2)^{-1/2}\left(\frac{t+a}{t-a}\right)^{i\beta} = \frac{\kappa-1}{\mu+1}\pi i. \tag{C7}$$

If we make the change of variable $t = -t$ in the first integral of (C7) we obtain

$$\int_a^\infty dt(t^2-a^2)^{-1/2}\left[\left(\frac{t+a}{t-a}\right)^{i\beta} - \left(\frac{t-a}{t+a}\right)^{i\beta}\right] = \frac{\kappa-1}{\kappa+1}\pi i, \tag{C8}$$

i.e.

$$\int_a^\infty dt(t^2-a^2)^{-1/2}\sin\beta\log\frac{t+a}{t-a} = \frac{\kappa-1}{\kappa+1}\frac{\pi}{2}$$

$$= \frac{\pi}{4}\frac{1-2\nu}{1-\nu}, \tag{C9}$$

from which (3.28) follows immediately.